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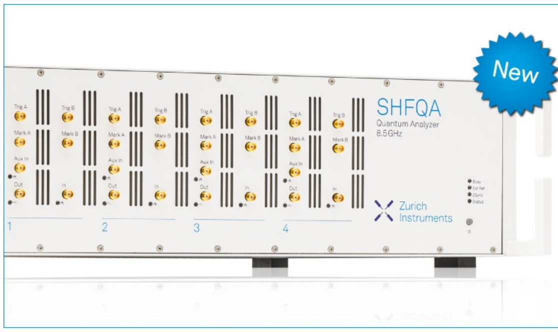

Exact Solution of the Convective Flow of a Viscous Fluid Layer with a Heated Lower Boundary

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Abstract. A new exact solution of the layered convection problem is considered. The obtained solution describes the flow layer of a viscous incompressible fluid with nonzero gradients of temperature and pressure. The horizontal velocity components depend only on the vertical transverse coordinate of the fluid layer. At the lower layer boundary, nonzero temperature gradients and the Navier slip condition are specified, tangential stresses and longitudinal pressure gradients being specified at the upper boundary. The possibility of the occurrence of counterflow areas and the corresponding changes in the tangential stresses and the vorticity vector are shown for the obtained particular exact solution.

PROBLEM STATEMENT

The system of equations including the Navier-Stokes equations in the Boussinesq approximation, the heat equation, and the incompressibility equation describes the convective flow of a viscous incompressible fluid in projections on the axes of the Cartesian coordinate system [1],

$$\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}=-\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}\right),$$

$$\frac{\partial V_y}{\partial t}+V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}+V_z\frac{\partial V_y}{\partial z}=-\frac{\partial P}{\partial y}+\nu\left(\frac{\partial^2 V_y}{\partial x^2}+\frac{\partial^2 V_y}{\partial y^2}+\frac{\partial^2 V_y}{\partial z^2}\right)$$

$$\frac{\partial V_z}{\partial t}+V_x\frac{\partial V_z}{\partial x}+V_y\frac{\partial V_z}{\partial y}+V_z\frac{\partial V_z}{\partial z}=-\frac{\partial P}{\partial z}+\nu\left(\frac{\partial^2 V_z}{\partial x^2}+\frac{\partial^2 V_z}{\partial y^2}+\frac{\partial^2 V_z}{\partial z^2}\right)+g\beta T, \quad (1)$$

$$\frac{\partial T}{\partial t}+V_x\frac{\partial T}{\partial x}+V_y\frac{\partial T}{\partial y}+V_z\frac{\partial T}{\partial z}=\chi\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right),$$

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}=0\,.$$

Here, V_x, V_y, and V_z are the velocities parallel to the corresponding coordinate axes of the Cartesian coordinate system $Oxyz$; $P=P(x,y,z)$ is the deviation of pressure from hydrostatic, taken relative to the constant average fluid density ρ; T is deviation from the average temperature; ν and χ are the coefficients of kinematic viscosity

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and thermal diffusivity of a fluid, respectively; g is gravity acceleration; β is the temperature coefficient of volume expansion.

The fluid layer flow is assumed to be steady and layered. Thus, the solution of the equation system (1) is sought in the following form [2–4]:

$$
\begin{gathered}
V_x = U(z),\ V_y = V(z),\ V_z = 0, \\
P = P_0(z) + xP_1(z) + yP_2(z), \\
T = T_0(z) + xT_1(z) + yT_2(z).
\end{gathered} \tag{2}
$$

We substitute the class of exact solutions (2) into the equation system (1). We use the method of indeterminate coefficients and obtain the following equation system:

$$
\begin{gathered}
T_1'' = 0,\ P_1' = g\beta T_1,\ T_2'' = 0,\ P_2' = g\beta T_2,\ \nu U'' = P_1, \\
\nu V'' = P_2,\ \chi T_0'' = UT_1 + VT_2,\ P_0' = g\beta T_0.
\end{gathered} \tag{3}
$$

THE EXACT SOLUTION OF THE BOUNDARY VALUE PROBLEM WITH THE NAVIER SLIP CONDITION AND A HEATED LOWER BOUNDARY

We find a particular exact solution for the equation system (3). We write the boundary conditions for the fluid layer under study. Assume that the lower boundary of the infinite horizontal fluid layer, specified by the equation $z = 0$, is absolutely solid and motionless. Heating and the Navier slip condition [5–9] are specified at the lower boundary. In view of the structure (2) of the selected generalized class of solutions, these conditions are written in the following form:

$$
\begin{gathered}
T_0(0) = 0,\ T_1(0) = A,\ T_2(0) = B, \\
\alpha \left.\frac{\partial U}{\partial z}\right|_{z=0} = U(0),\ \alpha \left.\frac{\partial V}{\partial z}\right|_{z=0} = V(0).
\end{gathered} \tag{4}
$$

Here, α is a dimensional slip factor (slip length).

The value of the pressure function is set on the upper (free) surface $z = h$. The temperature at the upper boundary is equal to the reference zero value. Besides, zero tangential stresses are specified at the upper boundary of the fluid layer. According to the structure of the solutions class (2), these conditions have the form

$$
\begin{gathered}
T_0(h) = T_1(h) = T_2(h) = 0, \\
P_0(h) = S_0,\ P_1(h) = S_1,\ P_2(h) = S_2, \\
\left.\frac{\partial U}{\partial z}\right|_{z=h} = 0,\ \left.\frac{\partial V}{\partial z}\right|_{z=h} = 0.
\end{gathered} \tag{5}
$$

The solution of the equations system (3) satisfying the boundary conditions (4), (5) is polynomial,

$$
U = \frac{S_1}{2\nu}\left[z^2 - 2h(z+\alpha)\right] + \frac{Ag\beta}{24h\nu}\left[-6h^2z^2 + 4hz^3 - z^4 + 4h^3(z+\alpha)\right],
$$

$$
V = \frac{S_2}{2\nu}\left[z^2 - 2h(z+\alpha)\right] + \frac{Bg\beta}{24h\nu}\left[-6h^2z^2 + 4hz^3 - z^4 + 4h^3(z+\alpha)\right],
$$

$$T_0 = \frac{(AS_1 + BS_2)}{120h\nu\chi}\left(2h^2 - 3hz + z^2\right)\left(4h^2 + 6hz - 3z^2 + 20h\alpha\right)z +$$

$$+\frac{\left(A^2 + B^2\right)g\beta}{1008h^2\nu\chi}\left(2h^2 - 3hz + z^2\right)\left[-4h^4 + 7h^2z^2 - 4hz^3 + z^4 - 2h^3\left(3z + 14\alpha\right)\right]z\,, \tag{6}$$

$$T_1 = A\left(1 - \frac{z}{h}\right),\ T_2 = B\left(1 - \frac{z}{h}\right),$$

$$P_0 = S_0 - \frac{g\beta\left(AS_1 + BS_2\right)}{240h\nu\chi}\left(h - z\right)^2\left[3h^4 + z^4 - 2hz^2\left(2z + 5\alpha\right) + 2h^3\left(3z + 5\alpha\right) + h^2z\left(z + 20\alpha\right)\right] +$$

$$+\frac{g^2\beta^2\left(A^2 + B^2\right)}{8064h^2\nu\chi}\left(h - z\right)^2\left[11h^6 + 15h^2z^4 - 6hz^5 + z^6 - 4h^3z^2\left(5z + 14\alpha\right) + h^5\left(22z + 56\alpha\right) + h^4z\left(z + 112\alpha\right)\right],$$

$$P_1 = S_1 - \frac{Ag\beta}{2h}\left(h - z\right)^2,\ P_2 = S_2 - \frac{Bg\beta}{2h}\left(h - z\right)^2.$$

It follows from the analysis of the formulas for the exact solution (6) that each of the velocity components is determined by the interaction of two streams: the stream caused by the pressure drop (the Poiseuille flow) and the stream caused by heating/cooling and the effect of the gravitational force (the thermogravitational flow). At a certain value of α, each of these flows separately may allow the appearance of a stagnation point. The overlap of these flows significantly complicates the topology of the velocity field.

The analysis of the obtained solution has shown that the velocity components U and V can have at most one stagnant point on the interval $(0;h)$ in view of the positive slip length α. Figure 1 shows the profiles of the functions U and V in the case of the existence of one zero point for each of the functions. Figure 2 depicts the V velocity vector hodograph illustrating the counterflow areas in the fluid layer under study.

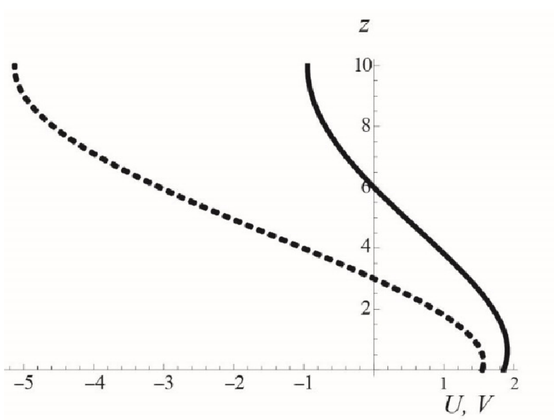

FIGURE 1. The profiles of the U (solid line) and V (dashed line) velocity components

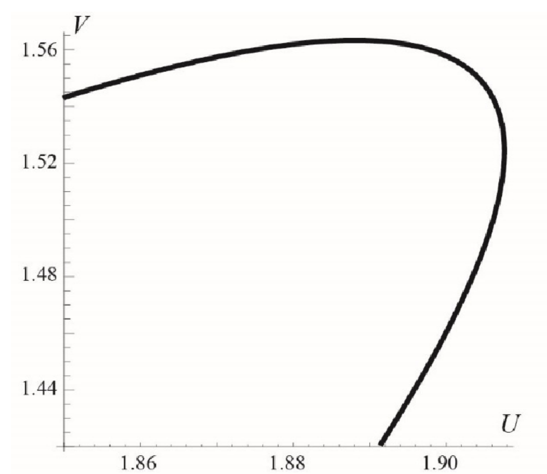

FIGURE 2. The hodograph of the velocity vector $\mathbf{V} = (U;V;0)$

For the obtained solution, we write the vorticity vector $\Omega = \left(\Omega_x, \Omega_y, \Omega_z\right)$ and the tangential stresses τ_{yz}, τ_{xz} arising in the flow of a viscous incompressible fluid. The components of the vorticity vector and of the tangential stress tensor are determined as follows:

$$\Omega_x = -\frac{\partial V_y}{\partial z} = -\tau_{yz}\,,\ \Omega_y = \frac{\partial V_x}{\partial z} = \tau_{xz}\,,\ \Omega_z = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\,. \tag{7}$$

Having substituted the obtained solution for the velocity components (6) into Eq. (7), we find that the vertical component of the vorticity vector is zero, $\Omega_z = 0$. The remaining components have the form

$$\Omega_x = \frac{S_2\left(h-z\right)}{\nu} - \frac{Bg\beta\left(h-z\right)^3}{6h\nu}\,,$$
$$\Omega_y = \frac{S_1\left(h-z\right)}{\nu} - \frac{Ag\beta\left(h-z\right)^3}{6h\nu}\,.$$

The vorticity components Ω_x and Ω_y in the fluid layer $z \in \left[0; h\right]$ can change their sign, and the corresponding tangential stresses can change from tensile to compressive, and vice versa (Fig. 3).

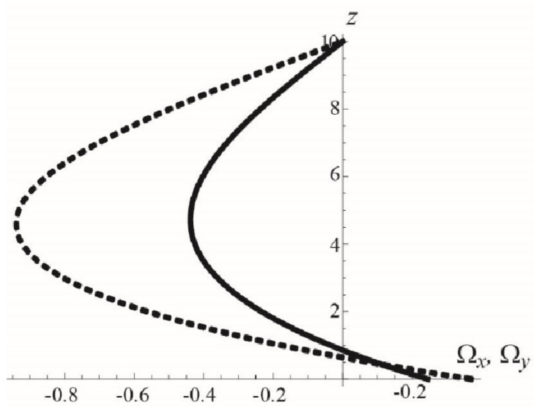

FIGURE 3. The profiles of the vorticity components Ω_y (solid line) and Ω_x (dashed line)

CONCLUSION

An exact solution has been obtained for the three-dimensional problem of the convective flow of a viscous incompressible fluid. The velocity components have been determined as functions of the transverse coordinate. The temperature and pressure in the fluid layer are set by linear functions in two longitudinal coordinates. The Navier slip condition and nonzero temperature gradients are specified on the lower solid surface of the fluid layer. On the upper free surface of the fluid layer, the tangential stresses and temperatures are set to be zero, and the pressure is considered to have constant, nonzero transverse gradients. It has been demonstrated how the method of determining the contact between a fluid and a solid surface affects the occurrence of stagnant points and counterflow areas in the flow of a fluid layer.

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